SS 2017

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Manifolds and Group actions

Homework 9

Mandatory Exercise 1. (5 points)

a) The Lie group $SO(3)$ acts on $S^2 \subset \mathbb{R}^3$. For any $A \in SO(3)$ denote by ϕ_A the map $\phi_A(p) = Ap$. Let $\eta \in \Omega^1(S^2)$ be a differential 1-form such that for any $A \in SO(3)$ it holds that $\phi_A^*\eta = \eta$. Show that $\eta = 0$. Hint: Take a point $p \in S^2$ and look only at those $A \in SO(3)$ which fix p, and at the equation $(\phi_A^*\eta)_p = \eta_p$. How does $(d\phi_A)_p$ act on the tangent space T_pS^2?

b) Part a) does not imply that any form ω that satisfies $\phi^*\omega = \omega$ vanishes. To see this, check that the formula

$$\omega_p(X, Y) = \langle p, X \times Y \rangle. \qquad X, Y \in T_pS^2,$$

defines a 2-form on S^2. Note that for $A \in SO(3)$ it holds that $(AX) \times (AY) = A(X \times Y)$. Hence this 2-form is invariant under the $SO(3)$ action and is nowhere zero.

Mandatory Exercise 2. (5 points)
Let V be a finite dimensional vector space. The unique possibe contraction on $V \otimes V^*$ is $c_{1,1} \colon V \otimes V^* \to \mathbb{R}$. Show that $c_{1,1}$ is the trace when one views $V \otimes V^*$ as $\mathrm{Lin}(V, V)$.

Mandatory Exercise 3. (10 points)
In the lecture the exterior derivative was defined using local coordinates. In this exercise we will give a coordinate independent definition. Let M be a smooth manifold and ω a k-form on M. Let $X_1, \ldots, X_{k+1}$ be smooth vector fields. We set

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) :=& \sum_{i=1}^{k+1} (-1)^{i-1} X_i \cdot \omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1}) + \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}), \end{aligned}$$

where the hat indicates an omitted variable.

a) Show that $d\omega$ is $C^\infty(M)$-linear, and conclude that the formula defines a $(k+1)$-form.

b) Let $\varphi : W \to \mathbb{R}^n$ be a coordinate system on M, and let $\omega = \sum_I a_I d\varphi_I$ be the local expression of ω. Show that the definition of the exterior derivitive in this exercise coincides with the definition of the exterior derivative given in the lectures, i.e. show that

$$d\omega = \sum_I da_I \wedge d\varphi_I.$$

Hand in on 26th of June in the pigeonhole on the third floor.